Pre-class Warm-up!!!

Which of the following statements is logically equivalent to the statement that the square matrix A is invertible?

a. The equation $Ax = 0$ is consistent

$Ax = 0$ is always consistent,

b. The reduced echelon form of A is the identity

$\Rightarrow$ we can compute A^{-1} by Gauss-Jordan elimination, so A is invertible.

$\Updownarrow$

c. The echelon form of A has a leading entry in every column

d. None of the above

e. More than one of the above.

If A is invertible then every equation $Ax = b$ has a solution: $x = A^{-1}b$

so the echelon form has a leading entry in every row. hence a leading entry in every column: C.

3.6 Determinants

Laplace expansion

$\shortparallel$

We learn:

- the definition in terms of cofactor expansions
- row and column properties, the effect under elementary operations
- computation using Gaussian elimination

New vocabulary:

- minors, cofactors, adjoint matrix, transpose matrix, upper triangular matrix

Some theorems:

- the formula for the inverse matrix using the adjoint matrix
- the determinant of the product is the product of the determinants
- Cramer's rule

Cofactors expansions of the determinant

Let $A = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \vdots \\ a_{31} & & \\ \cdot & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} 2 & 5 & 7 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$

The i,j minor of A is $M_{ij} = \det \begin{bmatrix} & & \\ & a_{ij} & \\ & & \end{bmatrix}$

where row i and column j are deleted.

We use $\det[a] = a$ $\quad \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

The i,j cofactor of A is $A_{ij} = (-1)^{i+j} M_{ij}$

e.g. $A_{11} = (-1)^{1+1} M_{11} = M_{11}$

$A_{32} = (-1)^{3+2} M_{32} = -M_{32} = 10$

$$\begin{bmatrix} + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ & \dots & & & & \end{bmatrix}$$

Cofactor (or Laplace) expansion along row i:

$\det A = |A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in}$

Along column j: $a_{1j}A_{1j} + \dots + a_{nj}A_{nj}$

Example: along row 3 $\quad$ The $(3,2)$ minor is $\det\begin{bmatrix} 2 & 7 \\ 2 & 2 \end{bmatrix} = 4 - 14 = -10$.

$\det\begin{bmatrix} 2 & 5 & 7 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} = 0 \cdot 10 - 1(-10) + 0(-10)$

$= 10$.

Row and column properties of the determinant

- cofactor expansions along any row or along any column are equal

The effect of elementary row operations on det:

- adding a multiple of a row to another row leaves the determinant unchanged
- switching two rows multiplies the determinant by -1
- multiplying a row by a number t multiplies the determinant by t .

Page 201 question 10:
Evaluate the determinant after simplifying by adding a multiple of some row or column to another.

$$\det \begin{bmatrix} -3 & 6 & 5 \\ 2 & -4 & 6 \\ 1 & -1 & 7 \end{bmatrix}$$

Solution. The det is unchanged by adding 2 · column 1 to column 2

We get $\begin{bmatrix} -3 & 0 & 5 \\ 2 & 0 & 6 \\ 1 & 1 & 7 \end{bmatrix}$

Calculate det using the col 2 expansion.

$$-0\begin{vmatrix} 2 & 6 \\ 1 & 7 \end{vmatrix} + 0\begin{vmatrix} -3 & 5 \\ 1 & 7 \end{vmatrix} - 1\begin{vmatrix} -3 & 5 \\ 2 & 6 \end{vmatrix}$$

$$= 28$$

- cofactor expansions along any row or along any column are equal

The effect of row operations:

- adding a multiple of a row to another row leaves the determinant unchanged
- switching two rows multiplies the determinant by -1
- multiplying a row by a number t multiplies the determinant by t .

Consequences

- If a matrix has a zero row or column then det = 0 from the Laplace expansion.
- If a matrix has two rows the same or two columns the same then det = 0
- The det of a triangular matrix is the product of the diagonal elements.

$$\det\begin{bmatrix} a & & ? & ? \\ & b & \ddots & \\ & & c & ? \\ O & & & \ddots \\ & & & h \end{bmatrix} = a \cdot b \cdot c \cdots h$$

Using 1st column expansion

$$\det\begin{bmatrix} & \end{bmatrix} = a\begin{vmatrix} b & & \cdots \\ & c & \\ O & & h \end{vmatrix} = a \cdot b \begin{vmatrix} c & ? \\ O & \ddots & \\ & & h \end{vmatrix}$$

$$\det\begin{bmatrix} \boxed{\quad v \quad} \\ \boxed{\quad v \quad} \end{bmatrix} = -\det\begin{bmatrix} \boxed{\quad v \quad} \\ \boxed{\quad v \quad} \end{bmatrix}$$

so $\det A = -\det A$, $\det A = 0$.

Like questions 13-16:
Use the method of elimination to evaluate the determinant.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 5 & 9 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Solution: $② \to ② - 5①$
$③ \to ③ - 2①$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -20 \\ 0 & -1 & -7 \end{bmatrix}$$

$③ \to ③ - ②$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -20 \\ 0 & 0 & 13 \end{bmatrix}$$

The determinant is unchanged and equals $1 \cdot (-1) \cdot 13 = -13$

$$B = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 9 & 0 \\ 1 & 2 & 4 \end{bmatrix}$$

Solution start with $① \leftrightarrow ③$, multiplies the det. by -1, so the answer is $+13$

Check $1 \cdot 9 \cdot 1 + 2 \cdot 0 \cdot 2 + 4 \cdot 5 \cdot 3$

$- 2 \cdot 9 \cdot 4 - 3 \cdot 0 \cdot 1 - 1 \cdot 5 \cdot 2$

$= -13$

Practical way to compute the determinant of a 3 x 3 matrix.

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - gec - hfa - idb$$

Pre-class Warm-up!!!

Question.
What are the determinants of the three elementary matrices

1. $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ det = 1

2. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ det = -1

3. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ det = 3

a. -1

b. 0

c. 1

d. 2

e. 3

More properties of the determinant

Defining properties:
det A is the unique function on square matrices for which

- it is linear on each row
- If we switch two rows, det A is multiplied by -1
- det (identity matrix) = 1

It has the same defining properties on columns

- If A, B are n x n matrices then
  det AB = det A det B
- The transpose matrix: det A^T = det A

Example $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

If we multiply a row by r, the det is multiplied by r.

If $A = \begin{bmatrix} \\ \boxed{\quad a_1 \quad} \\ \\ \end{bmatrix}$ $B = \begin{bmatrix} \\ \boxed{\quad b_1 \quad} \\ \\ \end{bmatrix}$

then $\det \begin{bmatrix} \\ \boxed{\quad a_1 + b_1 \quad} \\ \\ \end{bmatrix} = \det A + \det B.$

This relates to writing invertible matrices as products of elementary matrices.

More properties of the determinant

Theorem 5

If $\det A \neq 0$ then A is invertible and it's inverse is

$$A^{-1} = \frac{[A_{ij}]^T}{\det A}$$

where $A_{ij} = (-1)^{i+j} M_{ij}$ is the i,j cofactor

$[A_{ij}]^T$ is called the adjoint matrix.

Consequences: A square matrix A is invertible <=> $\det A \neq 0$

"$\Rightarrow$" If $AB = BA = I$ then

$\det(AB) = \det A \cdot \det B = \det I = 1$

so $\det A$ is invertible. "$\Leftarrow$" from Theorem 5.

Question 34

Find A^{-1} when $A = \begin{bmatrix} 2 & 0 & 3 \\ -5 & -4 & 2 \\ 2 & -1 & 1 \end{bmatrix}$

Solution: $\det A = 35$

$$[M_{ij}] = \begin{bmatrix} -2 & -9 & 13 \\ 3 & -4 & -2 \\ 12 & 19 & -8 \end{bmatrix}$$

$12 = \begin{vmatrix} 0 & 3 \\ -4 & 2 \end{vmatrix}$

$$[A_{ij}] = \begin{bmatrix} -2 & 9 & 13 \\ -3 & -4 & 2 \\ 12 & -19 & -8 \end{bmatrix}$$

$$A^{-1} = \frac{1}{35} \begin{bmatrix} -2 & -3 & 12 \\ 9 & -4 & -19 \\ 13 & 2 & -8 \end{bmatrix}$$

Cramer's rule

To solve $Ax = b$ where
$A = [c_1 \mid c_2 \mid \dots \mid c_n]$

eg $c_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ $c_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

$$x_i = \frac{\det [c_1 \mid \dots \mid b \mid \dots \mid c_n]}{\det A}$$

column c

Question: Solve

$$x + 2y = 1$$
$$4x + 3y = 2$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}} = \frac{-1}{-5} = \frac{1}{5}$$

$$y = \frac{\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}} = \frac{-2}{-5} = \frac{2}{5}$$